Outline: Curves on Surfaces

1. The Darboux Frame

Let S be a regular oriented surface, and let $\vec{x}(t)$ be a curve on S. The **Darboux frame** for $\vec{x}(t)$ at a point p consists of the following three vectors:

1. The unit normal vector $\vec{N}$ to the surface at p.
2. The unit tangent vector $\vec{T}$ the curve at p.
3. The third vector $\vec{U} = \vec{N} \times \vec{T}$, which is tangent to the surface and normal to the curve.

The three vectors $\{\vec{T}, \vec{U}, \vec{N}\}$ are a right-handed frame. When imagining a surface, we usually think of $\vec{N}$ as pointing towards us, in which case $\vec{U}$ is $90°$ counterclockwise from $\vec{T}$.

2. Geodesic and Normal Curvature

Let S be a regular oriented surface, and let $\vec{x}(t)$ be a curve on S. The **geodesic curvature** $\kappa_g(t)$ and **normal curvature** $\kappa_n(t)$ of $\vec{x}(t)$ are defined by the formula

$$\frac{d\vec{T}}{ds} = \kappa_n \vec{N} + \kappa_g \vec{U}.$$

That is,

$$\kappa_n = \vec{N} \cdot \frac{d\vec{T}}{ds} \qquad \text{and} \qquad \kappa_g = \vec{U} \cdot \frac{d\vec{T}}{ds}.$$

Note: Recall that $\dfrac{d\vec{T}}{ds} = \kappa \vec{P}$ for a space curve. This is often the easiest way to compute $\dfrac{d\vec{T}}{ds}$.

3. Normal Curvature and the Second Fundamental Form

Normal curvature is related to the second fundamental form. Specifically, if $\vec{x}(t)$ is any curve on a surface S, then

$$I\!I(\vec{T}) = \kappa_n$$

at each point on the curve, where $\vec{T}$ is the unit tangent vector to $\vec{x}(t)$, and κ_n is the normal curvature.

In particular, if $\vec{x}(t)$ is a curve whose unit tangent vector $\vec{T}$ points in one of the principle directions at a point p, then the normal curvature κ_n at p is equal to one of the principle curvatures of S at p. For example, on a surface of revolution, the principle curvatures are simply the normal curvatures of the usual coordinate lines.